Euler numbers of four-dimensional rotating black holes with the Euclidean signature

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Abstract

For a black hole's spacetime manifold in the Euclidean signature, its metric is positive definite and therefore a Riemannian manifold. It can be regarded as a gravitational instanton and a topological characteristic which is the Euler number is associated. In this paper we derive a formula for the Euler numbers of four-dimensional rotating black holes by the integral of the Euler density on the spacetime manifolds of black holes. Using this formula, we obtain that the Euler numbers of Kerr and Kerr-Newman black holes are 2. We also obtain that the Euler number of the Kerr-Sen metric in the heterotic string theory with one boost angle nonzero is 2 that is in accordance with its topology.

1 Introduction

Euler number is one of the topological invariants for Riemannian manifolds. It is the alternative sum of the Betti numbers, i.e., $\chi = \sum (-1)^p B_p$. For a black hole's spacetime manifold in the Euclidean signature its metric is positive definite and therefore is a Riemannian manifold. Therefore the Euler characteristic is associated with the manifold. The topologies of Kerr and Kerr-Newman black holes in the metrics of the maximal analytic extensions are $R^2 \times S^2$. The Euler number of a product manifold $M = M_1 \times M_2$ is the product of the Euler numbers of each manifold $\chi(M) = \chi(M_1) \times \chi(M_2)$. Thus one obtains that the Euler numbers for Kerr and Kerr-Newman black holes are 2 as well as other four-dimensional rotating black holes, such as those appear in the heterotic string theory [1-4]. The direct calculations for the Euler numbers of four-dimensional spherically symmetric black holes were given by the authors of Refs. [5-8]. The direct calculation for the Euler number of the Kerr metric was performed in Ref. [6]. However the direct calculation for the Euler number

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of the Kerr-Newman metric seems missing from the literature. In this paper we will derive a formula of Eq. (2.28) for the Euler numbers of four-dimensional rotating black holes using the Gauss-Bonnet formula. According to the formula (2.28), we can verify that the Euler numbers for the Kerr and Kerr-Newman metrics are 2. We can also obtain that the Euler number for the Kerr-Sen metric [1] in the heterotic string theory given by Sen is 2. Therefore the formula obtained in this paper for the Euler numbers of the four-dimensional black holes has some universality.

In this section we first write down the Gauss-Bonnet formula that is used to calculate the Euler numbers. Let M^n be a compact orientable Riemannian manifold of an even dimension n. According to Refs. [9,10] we define in M^n the intrinsic exterior differential form Ω of degree n which is equal to a scalar invariant of M^n multiplied by the volume element

$$\Omega = (-1)^{p-1} \frac{1}{2^{2p}\pi^p p!} \epsilon_{a_1 \cdots a_{2p}} \Omega^{a_1 a_2} \wedge \cdots \wedge \Omega^{a_{2p-1} a_{2p}} \ , \tag{1.1}$$

where $n = 2p$, $\epsilon_{a_1 \cdots a_{2p}}$ is a symbol which is equal to $+1$ or -1 according to i_1, $\cdots$, i_n to form an even or odd permutation of $1, \cdots, n$, and is zero otherwise. The unit vectors of the original manifold M^n form a larger and higher dimensional manifold M^{2n-1} of dimension $2n-1$. Chern [10] has shown that Ω is equal to the exterior derivative of a differential form Π of degree $n-1$ in M^{2n-1}:

$$\Omega = d\Pi \ . \tag{1.2}$$

Chern has obtained that the Euler number of the manifold can be expressed by the integral of the n-form Ω on the manifold:

$$\chi = \int_M \Omega \ . \tag{1.3}$$

Supposing that the definition of Π in the manifold M^{2n-1} can be extended to be defined in the original manifold M^n, then by using the Stokes theorem the integral of Eq. (1.3) can be converted to the integral of Π on the boundaries of the manifold. (In fact such extensions of the definition of Π in the original manifolds M^n are exist in the practical calculations such as the calculation of the four-dimensional case proceeded in the following of this paper.) According to Chern the manifold may be the compact orientable Riemannian. For compact manifolds they include compact manifolds with boundaries and compact manifolds without boundaries usually. For a four-dimensional spherically symmetric or rotating black hole, the horizon or outer horizon is a null surface which divides the spacetime into two parts. For an ideally permanent black hole, any information inside the horizon cannot escape out of the horizon. Therefore for the observer outside the horizon the area inside the horizon is unphysical. The physical area can be regarded to be surrounded by two three-dimensional hypersurfaces, one is the horizon or outer horizon, the other lies at infinity. The later can be realized through mapping the points of constant t, θ and ϕ with $r \to \infty$ onto a three-dimensional hypersurface at infinity. These two boundaries make the physical area of a black hole's spacetime be a compact orientable manifold with two boundaries. In their metrics of Euclidean signatures they are Riemannian. Therefore we can apply the Gauss-Bonnet formula (1.3) directly to calculate their Euler numbers and the integral areas can only be

taken outside the horizons. [1]

According to Eq. (1.1) the differential form of degree four which is the Euler density reads

$$\Omega = \frac{1}{32\pi^2}\epsilon_{abcd}\Omega^{ab} \wedge \Omega^{cd} \ . \tag{1.4}$$

From Eq. (1.3) Ω is equal to the exterior derivative of a differential form Π of degree three. To apply the Stokes theorem the integral of Eq. (1.3) can be carried out by the integral of Π on the manifold's boundaries. Then for a four-dimensional black hole we obtain

$$\chi = \int_{\partial M}\Pi = \int_\infty \Pi + \int_{r_h}\Pi \ . \tag{1.5}$$

In Eq. (1.5) the index ∞ means the boundary at infinity, r_h means the boundary at the horizon. (For a charged or a rotating black hole with two horizons, here and in the following we mean the horizon to be the outer horizon and take this for granted.) They are three-dimensional hypersurfaces. For the topological characteristic numbers of manifolds with boundaries generally one should consider certain boundary corrections. The index theorem is called the Atiyah-Patodi-Singer index theorem generally [9]. For the Euler number of a black hole's spacetime manifold, Eguchi, Gilkey, and Hanson [9] has given a boundary modification and the Gauss-Bonnet formula is given by

$$\chi = \frac{1}{32\pi^2}\int_V \epsilon_{abcd}\Omega^{ab} \wedge \Omega^{cd} - \frac{1}{32\pi^2}\int_{\partial M}\epsilon_{abcd}(2\theta^{ab} \wedge \Omega^{cd} - \frac{4}{3}\theta^{ab} \wedge \theta^c_{\ e} \wedge \theta^{ed}) \ , \tag{1.6}$$

where $\theta^{ab} = \omega^{ab} - (\omega_0)^{ab}$ is the second fundamental form of the boundary. This formula was used for the calculation of the Euler numbers of four-dimensional black holes in Refs. [5-8]. In this paper we will use Eqs. (1.3) and (1.5) directly to calculate the Euler numbers for four-dimensional black holes according to Chern [10]. At the end of this paper, we discuss the difference and connection between the method used in this paper and the method used in Refs. [5-8] for the calculation of the Euler numbers.

The Euler numbers for the extremal black holes need to be treated especially. For extremal black holes like those of the extremal Reissner-Nordström, Kerr and Kerr-Newman black holes, or the extremal black holes in the superstring theories, there are two cases according to the studies of many authors (see, e.g., Ref. [11]) due to their horizons are located

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[1]The reason why the integral area can only be taken outside the horizon of a black hole can also be understood from the point of combinatorial topology. From the sense of combinatorial topology, one divides a compact orientable manifold into many cells. The Euler number of the manifold is defined to be the alternative sum of the numbers of the cells of every dimension, i.e., $\chi = \sum(-1)^p d_p$, where d_p is the number of the cells of p-dimension under a certain division of the manifold. Supposing now one proceeding to take a practical division of a black hole's spacetime manifold, because the observer is outside the horizon, while any information inside the horizon cannot escape out of the horizon, the observer can not obtain the information for the division of the area inside the horizon. (Meantime, we think that an observer could not enter into and out again from the horizon of a black hole because of gravitation.) Therefore for the observer outside the horizon, the information of the division of the spacetime manifold that has sense is only of that outside the horizon. That is to say d_p is the number of the cells of p-dimension under a certain division of the manifold outside the horizon. This means that from the meaning of the differential topology as in Eq. (1.3), the integral area can only be taken outside the horizon.

at the infinities or finite r_h. If their horizons are located at infinities, then because the space-time metrics are asymptotically flat at infinities, the two terms in Eq. (1.5) are all tend to be zero. Therefore the Euler numbers are zero. This is in accordance with their topologies to be $S^1 \times R \times S^2$. If their horizons are located at finite r_h, then one can still use Eq. (1.5) to calculate their Euler numbers.

In Sec. II, we derive an expression of the Euler numbers for the general form of the metrics of four-dimensional rotating black holes. In Sec. III, we calculate the Euler numbers for several cases that includes Kerr, Kerr-Newman, and Kerr-Sen metrics. In Sec. IV, we discuss some of the problems and point out that the expression (2.28) obtained in this paper are not universal to all of the four-dimensional rotating black holes. In the Appendix, we derive the surface gravity of four-dimensional rotating black holes that is needed in the calculation of Sec. II.

2 The calculation

The metric of a four-dimensional rotating black hole is generally given by

$$ds^2 = -g_{tt}(r,\theta)dt^2 + g_{rr}(r,\theta)dr^2 + g_{\theta\theta}(r,\theta)d\theta^2 + g_{\phi\phi}(r,\theta)d\phi^2 - 2g_{t\phi}(r,\theta)dtd\phi \ , \qquad (2.1)$$

where in Eq. (2.1) the signature of the metric is Lorentzian. In the studies of the thermodynamical and topological properties of black holes one often needs to consider their metrics of the Euclidean signatures. The Euclidean metric of a rotating black hole can be obtained through Wick rotating both the time and angular momentum parameter of the Lorentzian metric, i.e., $t \rightarrow -i\tau$, $a \rightarrow ia$, where τ is the imaginary time [12,13]. The Euclidean form of the metric (2.1) can be written as

$$ds^2 = g_{\tau\tau}(r,\theta)d\tau^2 + g_{rr}(r,\theta)dr^2 + g_{\theta\theta}(r,\theta)d\theta^2 + g_{\phi\phi}(r,\theta)d\phi^2 + 2g_{\tau\phi}(r,\theta)d\tau d\phi \qquad (2.2)$$

generally. For the studies of the thermodynamical properties of a black hole, the Euclidean form metric makes a black hole be a finite temperature system [14,15]. The imaginary time τ is a periodic coordinate and all of the physical quantities like that of the Green functions are periodic with respect to the coordinate τ. For the studies of the topological properties of a black hole such as the Euler characteristic, because it is defined for a Riemannian manifold which is positive definite, one also needs to consider the Euclidean form metric with signature $(+ + + +)$. Thus the spacetime of a black hole is regarded as a total. The imaginary time τ does not have the time evaluation meaning any more. It is related with the thermodynamical properties of a black hole.

For the convenience of the calculation we write the metric (2.2) in the form

$$ds^2 = [a(r,\theta)d\tau + b(r,\theta)d\phi]^2 + c^2(r,\theta)dr^2 + f^2(r,\theta)d\theta^2 + h^2(r,\theta)d\phi^2 \ , \qquad (2.3)$$

where

$$a = \sqrt{g_{\tau\tau}} \ , \quad c = \sqrt{g_{rr}} \ , \quad f = \sqrt{g_{\theta\theta}} \ , \quad \sqrt{b^2 + h^2} = g_{\phi\phi} \ . \qquad (2.4)$$

Therefore we have

$$b = -\frac{g_{\tau\phi}}{a} \ , \qquad h = \sqrt{g_{\phi\phi} - \frac{g_{\tau\phi}^2}{g_{\tau\tau}}} \ . \tag{2.5}$$

The local orthogonal frame is defined as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{ab}e^a e^b \ , \tag{2.6}$$

where $e^a = e^a{}_\mu dx^\mu$. The spin connection one-form $\omega^a{}_b$ are determined uniquely by the Cantan's structure equations and the torsionless metric conditions

$$de^a + \omega^a{}_b \wedge e^b = 0 \ , \qquad \omega^a{}_b = -\omega^b{}_a = \omega^a{}_{b\mu}dx^\mu \ . \tag{2.7}$$

The solutions of Eq. (2.7) are

$$\omega^{01} = \frac{1}{c}\frac{\partial a}{\partial r}d\tau - \frac{1}{2c}\left(\frac{\partial b}{\partial r} + \frac{b}{a}\frac{\partial a}{\partial r}\right)d\phi \ , \tag{2.8a}$$

$$\omega^{02} = \frac{1}{f}\frac{\partial a}{\partial\theta}d\tau - \left(\frac{1}{2f}\frac{\partial b}{\partial\theta} + \frac{b}{2af}\frac{\partial a}{\partial\theta}\right)d\phi \ , \tag{2.8b}$$

$$\omega^{03} = \left(\frac{1}{2h}\frac{\partial b}{\partial\theta} - \frac{b}{2ah}\frac{\partial a}{\partial\theta}\right)d\theta + \left(\frac{1}{2h}\frac{\partial b}{\partial r} - \frac{b}{2ah}\frac{\partial a}{\partial r}\right)dr \ , \tag{2.8c}$$

$$\omega^{12} = \frac{1}{f}\frac{\partial c}{\partial\theta}dr - \frac{1}{c}\frac{\partial f}{\partial r}d\theta \ , \tag{2.8d}$$

$$\omega^{13} = \left(\frac{a}{2ch}\frac{\partial b}{\partial r} - \frac{b}{2ch}\frac{\partial a}{\partial r}\right)d\tau - \left(\frac{1}{c}\frac{\partial h}{\partial r} + \frac{b}{2ch}\frac{\partial b}{\partial r} - \frac{b^2}{2ach}\frac{\partial a}{\partial r}\right)d\phi \ , \tag{2.8e}$$

$$\omega^{23} = -\left(\frac{b}{2fh}\frac{\partial a}{\partial\theta} - \frac{a}{2fh}\frac{\partial b}{\partial\theta}\right)d\tau - \left(\frac{1}{f}\frac{\partial h}{\partial\theta} + \frac{b}{2fh}\frac{\partial b}{\partial\theta} - \frac{b^2}{2afh}\frac{\partial a}{\partial\theta}\right)d\phi \ . \tag{2.8f}$$

The curvature two-form Ω^a_b are determined by the following equations

$$\Omega^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \ . \tag{2.9}$$

Then Eq. (1.4) is evaluated to be

$$\frac{1}{32\pi^2}\epsilon_{abcd}\Omega^{ab} \wedge \Omega^{cd} = \frac{1}{4\pi^2}(\Omega^{01} \wedge \Omega^{23} + \Omega^{20} \wedge \Omega^{13} + \Omega^{03} \wedge \Omega^{12}) \ . \tag{2.10}$$

For the metric (2.3) we obtain

$$\Omega^{01} \wedge \Omega^{23} + \Omega^{20} \wedge \Omega^{13} + \Omega^{03} \wedge \Omega^{12} = -d(\omega^{01} \wedge \omega^{02} \wedge \omega^{03}) - d(\omega^{01} \wedge \omega^{12} \wedge \omega^{13})$$

$$-d(\omega^{02} \wedge \omega^{12} \wedge \omega^{23}) - d(\omega^{03} \wedge \omega^{13} \wedge \omega^{23}) + d(\omega^{01} \wedge d\omega^{23})$$

$$+d(\omega^{13} \wedge d\omega^{02}) + d(\omega^{03} \wedge d\omega^{12}) \ . \tag{2.11}$$

According to Eq. (1.2) we have

$$d\Pi = \frac{1}{4\pi^2}(\Omega^{01} \wedge \Omega^{23} + \Omega^{20} \wedge \Omega^{13} + \Omega^{03} \wedge \Omega^{12}) . \tag{2.12}$$

Therefore from Eqs. (2.11) and (2.12) we can read out the solution of Π. However this is not necessary for the calculation of the Euler numbers. The reason is that the Euler number χ is expressed by Eq. (1.5) in the form of the integral on the three-dimensional hypersurfaces at the horizon and infinity. Therefore we only need to extract the three-form of $d\tau \wedge d\theta \wedge d\phi$ terms from Eqs. (2.11) and (2.12).

To calculate the Euler number for the metric (2.2) directly is rather complicated. However we find that it will be simplified to make a rotating coordinate transformation

$$\phi' = \phi - \Omega_h \tau , \tag{2.13}$$

where Ω_h is the angular velocity of the outer horizon and so is a constant. Because the Euler number is a diffeomorphism invariant quantity it is invariant under a diffeomorphism transformation. And we know that an infinitesimal rotating transformation is a diffeomorphism transformation and a finite rotation can be obtained through many infinitesimal rotating transformations. Therefore the Euler number for a rotating black hole is invariant under the rotating (2.13). After the rotating (2.13) the metric (2.2) becomes

$$ds'^{\,2} = G_{\tau\tau}d\tau^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi'^{\,2} + 2g'_{\tau\phi}d\tau d\phi' , \tag{2.14}$$

where

$$G_{\tau\tau} = g_{\tau\tau} + 2g_{\tau\phi}\Omega_h + g_{\phi\phi}\Omega_h^2 , \qquad g'_{\tau\phi} = g_{\phi\phi}\Omega_h + g_{\tau\phi} . \tag{2.15}$$

On the horizon the three-dimensional metric is diagonal because

$$g'_{\tau\phi}|_{r=r_h} = 0 . \tag{2.16}$$

The reason is that for a four-dimensional rotating black hole its angular velocity on the horizon is $-g_{\tau\phi}/g_{\phi\phi}|_{r=r_h}$. We also have

$$G_{\tau\tau}|_{r=r_h} = 0 . \tag{2.17}$$

This can be seen clearly in the Appendix. We can see that under the rotating coordinate transformation (2.13) g_{rr}, $g_{\theta\theta}$, and $g_{\phi\phi}$ keep unchanged. g^{rr} is not changed either. As null surfaces the horizons for the metrics (2.2), (2.3), and (2.14) are determined by $g^{rr} = 0$. Therefore under the coordinate transformation (2.13) the horizons of the black holes are not changed with respect to their original locations. Because on the horizon $G_{\tau\tau} = 0$, the horizons are also determined by $G_{\tau\tau} = 0$. On the other hand the stationary limit surfaces for the metric (2.14) are determined by $G_{\tau\tau} = 0$, it is the same equation to determine the horizons. Therefore for the metric (2.14) the stationary limit surfaces are coinciding with the horizons.

Now for the metric (2.14) it can still be incorporated in the form of Eqs. (2.3) to (2.5), where now

$$a = \sqrt{G_{\tau\tau}} \;, \qquad c = \sqrt{g_{rr}} \;, \qquad f = \sqrt{g_{\theta\theta}} \;, \qquad \sqrt{b^2 + h^2} = g_{\phi\phi} \;. \tag{2.18}$$

And we have

$$b = -\frac{g'_{\tau\phi}}{a} \;, \qquad h = \sqrt{g_{\phi\phi} - \frac{g'^2_{\tau\phi}}{g_{\tau\tau}}} \;. \tag{2.19}$$

Because on the horizon $g'_{\tau\phi} = 0$ we have

$$b(r_h, \theta) = 0 \;, \qquad \frac{\partial b(r_h, \theta)}{\partial\theta} = 0 \;, \qquad h(r_h, \theta) = \sqrt{g_{\phi\phi}(r_h, \theta)} \;. \tag{2.20}$$

However on the horizon $\partial b(r, \theta)/\partial r$ may not be zero. From Eq. (1.5) the Euler number χ is expressed by the integral on the three-dimensional hypersurfaces at the horizon and infinity. The integrands are $d\tau \wedge d\theta \wedge d\phi$ three-forms extracted from Eqs. (2.11) and (2.12). For spherically and axially symmetric black holes, the spacetime metrics are asymptotically flat at infinities. We can see that in Eq. (1.5) the integrals at infinities tend to be zero. Explicit verifications for this fact are omitted here. Therefore only the integrals on the horizons are left in Eq. (1.5).

According to Eqs. (2.8) to (2.12) and (2.20) we can write down the non-zero three-form of $d\tau \wedge d\theta \wedge d\phi$ terms extracted from Π. They are the sum of the following four parts:

$$\Pi_1 = -\frac{1}{c}\frac{\partial f}{\partial r}\left[\frac{1}{c^2}\frac{\partial a}{\partial r}\frac{\partial h}{\partial r} - \frac{a}{8c^2 h}\left(\frac{\partial b}{\partial r}\right)^2\right] \;, \qquad \Pi_2 = -\frac{1}{c}\frac{\partial f}{\partial r}\frac{1}{f^2}\frac{\partial a}{\partial\theta}\frac{\partial h}{\partial\theta} \;,$$

$$\Pi_3 = -\frac{1}{c}\frac{\partial a}{\partial r}\frac{\partial}{\partial\theta}\left(\frac{1}{f}\frac{\partial h}{\partial\theta}\right) \;, \qquad \Pi_4 = \frac{1}{c}\frac{\partial h}{\partial r}\frac{\partial}{\partial\theta}\left(\frac{1}{f}\frac{\partial a}{\partial\theta}\right) \;. \tag{2.21}$$

Now the Euler number is given by the following integral:

$$\begin{aligned}
\chi &= \frac{1}{4\pi^2}\int_0^\beta d\tau \int_0^{2\pi} d\phi \int_0^\pi d\theta(\Pi_1 + \Pi_2 + \Pi_3 + \Pi_4)|_{r=r_h} \\
&= \frac{\beta}{2\pi}\int_0^\pi d\theta(\Pi_1 + \Pi_2 + \Pi_3 + \Pi_4)|_{r=r_h} \;,
\end{aligned} \tag{2.22}$$

where β is the inverse temperature. The Euclidean time in the integral takes one period because it is the periodic coordinate. According to Eqs. (A5) and (2.18) $a = \sqrt{G_{\tau\tau}}$ is zero on the horizon, therefore we have

$$\left.\frac{\partial a}{\partial\theta}\right|_{r=r_h} = 0 \;. \tag{2.23}$$

Thus in Eq. (2.22) $\Pi_2(r_h, \theta) = \Pi_4(r_h, \theta) = 0$. Therefore we obtain

$$\chi = \frac{\beta}{2\pi}\int_0^\pi d\theta(\Pi_1 + \Pi_3)|_{r=r_h} \;. \tag{2.24}$$

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According to the black hole thermodynamics, the inverse temperature β is equal to $2\pi/\kappa$, where κ is the surface gravity on the horizon. In the Appendix we derive a formula of Eq. (A12) for the surface gravity for a four-dimensional rotating black hole which is needed in the calculation of Eq. (2.24). In the Appendix we can also see that the surface gravity κ is invariant under general coordinate transformations, such as the rotating of Eq. (2.13). In the form of Eqs. (2.3), (2.18), and Eq. (A12) we have

$$\kappa = \frac{1}{c}\frac{\partial a}{\partial r}\bigg|_{r=r_h} \,, \qquad \beta = \frac{2\pi c}{\partial_r a}\bigg|_{r=r_h} \,. \tag{2.25}$$

To insert Eq. (2.25) into Eq. (2.24) and to move β inside the integral because the surface gravity κ is a constant on the horizon as well as β we obtain

$$
\begin{aligned}
\chi &= \int_0^\pi d\theta \frac{c}{\partial_r a}\left[-\frac{1}{c}\frac{\partial f}{\partial r}\left(\frac{1}{c^2}\frac{\partial a}{\partial r}\frac{\partial h}{\partial r} - \frac{a}{8c^2 h}\frac{\partial b}{\partial r}\frac{\partial b}{\partial r}\right) - \frac{1}{c}\frac{\partial a}{\partial r}\frac{\partial}{\partial\theta}\left(\frac{1}{f}\frac{\partial h}{\partial\theta}\right)\right]\bigg|_{r=r_h} \\
&= -\int_0^\pi d\theta \left[\frac{1}{c^2}\frac{\partial f}{\partial r}\left(\frac{\partial h}{\partial r} - \frac{a}{8c^2 h\partial_r a}\frac{\partial b}{\partial r}\frac{\partial b}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{f}\frac{\partial h}{\partial\theta}\right)\right]\bigg|_{r=r_h} \,.
\end{aligned}
$$

$$\tag{2.26}$$

Because $1/c^2 = g^{rr}$, the horizons of the metrics (2.3) and (2.14) are determined by $g^{rr}(r_h, \theta) = 0$, the first two terms in Eq. (2.26) vanish and we have

$$\chi = -\int_0^\pi d\theta \frac{\partial}{\partial\theta}\left(\frac{1}{f(r_h,\theta)}\frac{\partial h(r_h,\theta)}{\partial\theta}\right) \,, \tag{2.27}$$

where $f = \sqrt{g_{\theta\theta}}$ and $h = \sqrt{g_{\phi\phi}}$ from Eq. (2.20). At last we can write Eq. (2.27) in the form

$$\chi = -\int_0^\pi d\theta \frac{1}{2\sqrt{g_{\theta\theta}g_{\phi\phi}}}\left[g''_{\phi\phi} - \frac{1}{2g_{\phi\phi}}g'^{\,2}_{\phi\phi} - \frac{1}{2g_{\theta\theta}}g'_{\theta\theta}g'_{\phi\phi}\right]\bigg|_{r=r_h} \,, \tag{2.28}$$

where the prime is the derivative with respect to θ. According to this formula, the Euler number for a four-dimensional spherically symmetric or rotating black hole is determined by $g_{\theta\theta}$ and $g_{\phi\phi}$ in their Euclidean metric. However whether this formula is universal to all of the four-dimemsional rotating black holes will be discussed in Sec. IV.

3   Some examples

In this section we will take some examples for the above derived formulas (2.27) and (2.28). First we take a look at the four-dimensional spherically symmetric black holes. Their metrics in the Euclidean form are

$$ds^2 = e^{2U(r)}d\tau^2 + e^{-2U(r)}dr^2 + R^2(r)(d^2\theta + \sin^2\theta d\phi^2) \,. \tag{3.1}$$

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We can obtain that their Euler numbers are 2 from Eq. (2.27) directly. This result is also held for the four-dimensional spherically symmetric black holes appear in the superstring theories [16-19]. For the extremal four-dimensional spherically symmetric black holes such as the extremal Reissner-Nordström black hole, if their horizons are located at infinities along spacelike directions like that pointed out by Hawking *et al.* [20], their Euler numbers are zero according to Eq. (1.5) and their entropies are also zero. If their horizons are located at finite r_h like that discussed in Ref. [11], their Euler numbers are still 2 according to Eq. (2.27) and their entropies are still $A_h/4$. For the extremal black holes in the superstring theories they are similar as that of the extremal Reissner-Nordström black hole. If there are Yang-Mills fields in the Einstein's gravitational theory, for the four-dimensional spherically symmetric black holes their metrics may be modified to be [21,22]

$$ds^2 = e^{2U(r)}e^{-2\delta(r)}d\tau^2 + e^{-2U(r)}dr^2 + R^2(r)(d^2\theta + \sin^2\theta d\phi^2) , \qquad (3.2)$$

where $e^{-2\delta(r)}$ are the corrections due to Yang-Mills fields. We can still obtain that their Euler numbers are 2 according to Eq. (2.27).

For the Kerr black hole its metric in the Lorentzian form is

$$ds^2 = -\left(1 - \frac{2Mr}{r^2 + a^2\cos^2\theta}\right)dt^2 + \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2 - 2Mr}dr^2 + (r^2 + a^2\cos^2\theta)d\theta^2$$
$$+ \left[(r^2 + a^2)\sin^2\theta + \frac{2Mra^2\sin^4\theta}{r^2 + a^2\cos^2\theta}\right]d\phi^2 - \frac{4Mra\sin^2\theta}{r^2 + a^2\cos^2\theta}dtd\phi ,$$

$$(3.3)$$

where M is the total mass, a is the angular momentum per unit mass. Through Wick rotating the time and the angular momentum parameter one can obtain the Euclidean form metric. In Ref. [12] it is given by

$$ds^2 = V\left[d\tau - 2\frac{aMr\sin^2\theta}{\Delta + a^2\sin^2\theta}d\phi\right]^2$$
$$+ \frac{1}{V}\left[\frac{\Delta + a^2\sin^2\theta}{\Delta}dr^2 + (\Delta + a^2\sin^2\theta)d\theta^2 + \Delta\sin^2\theta d\phi^2\right] ,$$

$$(3.4)$$

where

$$V = 1 - \frac{2Mr}{r^2 - a^2\cos^2\theta} , \qquad \Delta = r^2 - 2Mr - a^2 . \qquad (3.5)$$

To write Eq. (3.4) in the explicit form it is

$$ds^2 = \left(1 - \frac{2Mr}{r^2 - a^2\cos^2\theta}\right)d\tau^2 + \frac{r^2 - a^2\cos^2\theta}{r^2 - a^2 - 2Mr}dr^2 + (r^2 - a^2\cos^2\theta)d\theta^2$$
$$+ \left[(r^2 - a^2)\sin^2\theta - \frac{2Mra^2\sin^4\theta}{r^2 - a^2\cos^2\theta}\right]d\phi^2 - \frac{4Mra\sin^2\theta}{r^2 - a^2\cos^2\theta}d\tau d\phi .$$

To compare Eq. (3.3) with Eq. (3.6) we can see that except the change of the signature and $t \to \tau$, the only other change is $a^2 \to -a^2$. All other forms of the metric coefficients are the same. Therefore to insert $g_{\theta\theta}$ and $g_{\phi\phi}$ of Eq. (3.6) into Eq. (2.28) and to note that $r_h^2 - a^2 - 2Mr_h = 0$ on the horizon at last we obtain

$$
\begin{aligned}
\chi &= 4M^2 r_h^2 \int_0^\pi \frac{r_h^2 + 3a^2 \cos^2\theta}{(r_h^2 - a^2 \cos^2\theta)^3} \sin\theta d\theta \\
&= (r_h^2 - a^2)^2 \int_0^\pi \frac{r_h^2 + 3a^2 \cos^2\theta}{(r_h^2 - a^2 \cos^2\theta)^3} d(-\cos\theta) .
\end{aligned}
$$

(3.7)

Let $\cos\theta = x$, we can obtain that the result of the integral is $2/(r_h^2 - a^2)^2$. Therefore we have

$$
\chi_{\mathrm{Kerr}} = 2 .
$$

(3.8)

This result was previously obtained by Liberati and Pollifrone [6]. Here we obtain it from a different method.

The metric of the Kerr-Newman black hole in the Lorentzian form is

$$
\begin{aligned}
ds^2 = &- \left(1 - \frac{2Mr - Q^2}{r^2 + a^2 \cos^2\theta}\right) dt^2 + \frac{r^2 + a^2 \cos^2\theta}{r^2 + a^2 - 2Mr + Q^2} dr^2 + (r^2 + a^2 \cos^2\theta) d\theta^2 \\
&+ \left[(r^2 + a^2)\sin^2\theta + \frac{(2Mr - Q^2)a^2 \sin^4\theta}{r^2 + a^2 \cos^2\theta}\right] d\phi^2 - \frac{2(2Mr - Q^2)a \sin^2\theta}{r^2 + a^2 \cos^2\theta} dt d\phi ,
\end{aligned}
$$

(3.9)

where M is the total mass, a is the angular momentum per unit mass, and Q is the electric charge. Like that of the Kerr metric, one can obtain its Euclidean form metric to be

$$
\begin{aligned}
ds^2 = &\left(1 - \frac{2Mr - Q^2}{r^2 - a^2 \cos^2\theta}\right) d\tau^2 + \frac{r^2 - a^2 \cos^2\theta}{r^2 - a^2 - 2Mr + Q^2} dr^2 + (r^2 - a^2 \cos^2\theta) d\theta^2 \\
&+ \left[(r^2 - a^2)\sin^2\theta - \frac{(2Mr - Q^2)a^2 \sin^4\theta}{r^2 - a^2 \cos^2\theta}\right] d\phi^2 - \frac{2(2Mr - Q^2)a \sin^2\theta}{r^2 - a^2 \cos^2\theta} d\tau d\phi
\end{aligned}
$$

(3.10)

through Wick rotating the time and the angular momentum parameter. The differences between Eq. (3.9) and Eq. (3.10) are the signature, $t \to \tau$, and $a^2 \to -a^2$. All other forms of the metric coefficients are the same. To insert $g_{\theta\theta}$ and $g_{\phi\phi}$ of Eq. (3.10) into Eq. (2.28) we obtain

$$
\chi = (2Mr_h - Q^2)^2 \int_0^\pi \frac{r_h^2 + 3a^2 \cos^2\theta}{(r_h^2 - a^2 \cos^2\theta)^3} \sin\theta d\theta .
$$

(3.11)

Eq. (3.11) is exactly Eq. (3.7) except that $2Mr_h$ is replaced by $2Mr_h - Q^2$. To note that on the horizon $r_h^2 - a^2 - 2Mr_h + Q^2 = 0$, therefore we have

$$\chi_{\text{Kerr-Newman}} = 2 \ . \tag{3.12}$$

The direct calculation for the Euler number of the Kerr-Newman black hole seems missing from the literatures. Here we give an exact calculation for this problem. For the extremal Kerr and Kerr-Newman black holes, there are two cases due to their horizons are located at the infinities or finite r_h like that of the Reissner-Nordström black hole. If their horizons are located at infinities, their Euler numbers are zero according to Eq. (1.5) for the reason that their metrics are asymptotically flat at infinities, and their entropies are also zero. If their horizons are located at finite r_h, their Euler numbers are still calculated as the above and therefore are 2, their entropies are still $A_h/4$ [23].

Next we will consider a four-dimensional rotating black hole in the superstring theories given by Sen [1]. It is a solution of the classical equations of motion of the low-energy effective field theory in the heterotic stirng theory. The Einstein canonical metric in the Lorentzian signature is given by

$$
\begin{aligned}
ds_E^2 = \ & - \ \frac{r^2 + a^2 \cos^2\theta - 2mr}{r^2 + a^2 \cos^2\theta + 2mr \sinh^2(\alpha/2)} dt^2 + \frac{r^2 + a^2 \cos^2\theta + 2mr \sinh^2(\alpha/2)}{r^2 + a^2 - 2mr} dr^2 \\
& + \ [r^2 + a^2 \cos^2\theta + 2mr \sinh^2(\alpha/2)]d\theta^2 - \frac{4mra \cosh^2(\alpha/2)\sin^2\theta}{r^2 + a^2 \cos^2\theta + 2mr \sinh^2(\alpha/2)} dt d\phi \\
& + \ \{(r^2 + a^2)(r^2 + a^2 \cos^2\theta) + 2mra^2 \sin^2\theta + 4mr(r^2 + a^2)\sinh^2(\alpha/2) \\
& \quad + 4m^2 r^2 \sinh^4(\alpha/2)\} \times \frac{\sin^2\theta}{r^2 + a^2 \cos^2\theta + 2mr \sinh^2(\alpha/2)} d\phi^2 \ .
\end{aligned}
\tag{3.13}
$$

It describes a four-dimensional rotating black hole with mass M, charge Q, angular momentum J, and magnetic dipole moment μ given by

$$M = \frac{m}{2}(1 + \cosh\alpha) \ , \qquad Q = \frac{m}{\sqrt{2}}\sinh\alpha \ ,$$

$$J = \frac{ma}{2}(1 + \cosh\alpha) \ , \qquad \mu = \frac{1}{\sqrt{2}}ma \sinh\alpha \ . \tag{3.14}$$

The dilaton field for the solution is

$$\Phi = -\ln \frac{r^2 + a^2 \cos^2\theta + 2mr \sinh^2(\alpha/2)}{r^2 + a^2 \cos^2\theta} \ . \tag{3.15}$$

Its string σ-model metric is given by

$$ds^2 = e^{-\Phi}ds_E^2 \ . \tag{3.16}$$

To transform to the Euclidean signature is the same as the above cases. The changes are the signature, $t \to \tau$, and $a^2 \to -a^2$. All other forms of the metric coefficients are the same.

The horizons for the metric (3.13) are determined by $g^{rr} = 0$. For its Euclidean signature metric we have on the horizon $r_h^2 - a^2 - 2mr_h = 0$. To insert $g_{\theta\theta}$ and $g_{\phi\phi}$ of its Euclidean signature metric into Eq. (2.28) we obtain finally

$$\chi = 4m^2 r_h^2 \cosh^4(\alpha/2) \int_0^\pi \frac{r_h^2 + 2mr_h \sinh^2(\alpha/2) + 3a^2 \cos^2\theta}{(r_h^2 + 2mr_h \sinh^2(\alpha/2) - a^2 \cos^2\theta)^3} d(-\cos\theta) \ . \qquad (3.17)$$

The result of the integral is calculated to be $2/4m^2 r_h^2 \cosh^4(\alpha/2)$. Therefore we obtain

$$\chi_{\text{Kerr-Sen}} = 2 \ . \qquad (3.18)$$

Therefore we give an exact calculation for the Euler number of the Kerr-Sen metric. It is in accordance with its topology to be $R^2 \times S^2$.

4 Discussion

In this paper we derive a formula (2.28) for the Euler numbers of four-dimensional rotating black holes using the Gauss-Bonnet formula through integrating the Euler density on a black hole's spacetime manifold outside the horizon. From this formula we obtain the correct results for the Euler numbers for many cases. For Kerr and Kerr-Newman metrics it can be verified that their Euler numbers are 2. It also stands for the Kerr-Sen metric of the four-dimensional rotating black hole in the heterotic string theory with one boost angle nonzero [1]. However this not means that the formula (2.28) for the Euler numbers is universal for all of the four-dimensional rotating black holes.

For the other known four-dimensional rotating black holes such as the rotating dilaton black hole solution of Ref. [24], to use Eq. (2.28) we can not obtain its Euler number to be 2. Only for its slowly rotating case - the metric (31) of Ref. [24], we can still obtain $\chi = 2$ from Eq. (2.28). For the four-dimensional rotating heterotic string black holes generated from the Kerr solution with more than one boost angle as those of Refs. [2-4] or the Anti-de Sitter rotating black hole [13,25], we can not obtain the correct results $\chi = 2$ either from Eq. (2.28). However for all these four-dimensional rotating black holes, their Euler numbers are 2 because their topologies are $R^2 \times S^2$. This means that for these cases we should consider proper boundary modifications like that of Eq. (1.6) to obtain their Euler numbers correctly.

At the end of this paper, it is also useful to point out the difference between the method used in the Introduction of this paper and the method used in Refs. [5-8] for the calculation of the Euler numbers. In this paper as we asserted in the Introduction, the integral area of the Euler density Ω is taken to be the area outside the horizon. Therefore the manifold of a black hole is treated as a compact manifold surrounded by two boundaries, one is the horizon and the other lies at the infinity. Therefore we do not consider again the boundary corrections of the Euler numbers which is the second term of Eq. (1.6). In Refs. [5-8], the integral areas of the Euler density Ω are also include the areas inside the horizons, therefore there need to consider the boundary corrections on the horizons, which is the second term of Eq. (1.6) that is obtained by Eguchi, Gilkey, and Hanson [9]. In fact it is just the boundary

corrections on the horizons that give the correct results of the Euler numbers in Refs. [5-8]. Therefore we think that the method used in this paper is equivalent to the method used in Refs. [5-8] in fact. Therefore we doubt that for other four-dimensional rotating black holes such as those of Refs. [2-4,24,25], there may need some other boundary corrections other than the second term of Eq. (1.6) to obtain the correct results of the Euler numbers.

APPENDIX: SURFACE GRAVITIES FOR ROTATING BLACK HOLES

In this appendix we give an explanation for Eqs. (2.17) and (2.25) for a four-dimensional rotating black hole. The metric for a four-dimensional rotating black hole is given by Eq. (2.2) generally in the Euclidean form. For the metric (2.2) there exists the following Killing field

$$\xi^\mu = \frac{\partial}{\partial \tau} + \Omega_h \frac{\partial}{\partial \phi} \ , \tag{A1}$$

where Ω_h is the angular velocity of the horizon which is a constant. Here we mean the horizon to be the outer horizon for a rotating black hole. Because the horizon is defined to be a null surface and ξ^μ is normal to the horizon, we have on the horizon [26]

$$\xi^\mu \xi_\mu |_{r=r_h} = 0 \ . \tag{A2}$$

For the metric of Eq. (2.2) we have

$$\xi^\mu \xi_\mu = g_{\tau\tau} + 2g_{\tau\phi}\Omega_h + g_{\phi\phi}\Omega_h^2 \ . \tag{A3}$$

For convenience we define

$$G_{\tau\tau} = g_{\tau\tau} + 2g_{\tau\phi}\Omega_h + g_{\phi\phi}\Omega_h^2 \ . \tag{A4}$$

Therefore from Eq. (A2) we have

$$G_{\tau\tau}|_{r=r_h} = 0 \ . \tag{A5}$$

To adopt Wald's symbol of Ref. [26] we write Eq. (A3) as

$$\xi^\mu \xi_\mu = -\lambda^2 \tag{A6}$$

in the spacetime of the black hole where λ is a scalar function and is a constant on the horizon. According to Eq. (A3), $\lambda^2 = -G_{\tau\tau}$ in fact for the metric (2.2) for a four-dimensional rotating black hole. Let ∇^μ represent the covariant derivative operator, thus $\nabla^\mu(\xi^\nu \xi_\nu)$ is also normal to the horizon. According to Refs. [26,27] there exists a function κ such that

$$\nabla^\mu(-\lambda^2) = -2\kappa\xi^\mu \ , \tag{A7}$$

where on the horizon $\kappa(r_h)$ is a constant and just the horizon's surface gravity.

Samely we can set up the lower index equation

$$\nabla_\mu(-\lambda^2) = -2\kappa\xi_\mu \ . \tag{A8}$$

The product of Eqs. (A7) and (A8) results

$$\nabla^\mu(\lambda^2)\nabla_\mu(\lambda^2) = -4\kappa^2\lambda^2 . \tag{A9}$$

Because λ^2 is a scalar function, κ^2 is also a scalar function. Therefore the surface gravity is invariant under the general coordinate transformations, such as the rotating coodinate transformation of Eq. (2.13). From Eqs. (A3), (A4), (A6), (A9), and the axially symmetric of the metric we obtain

$$4\kappa^2 G_{\tau\tau} = g^{rr}(\partial_r G_{\tau\tau})^2 + g^{\theta\theta}(\partial_\theta G_{\tau\tau})^2 . \tag{A10}$$

Because of Eq. (A5) we have

$$\lim_{r\to r_h} \partial_\theta G_{\tau\tau} = 0 . \tag{A11}$$

Therefore to take the limit $r \to r_h$ in both sides of Eq. (A10) results

$$\kappa(r_h) = \lim_{r\to r_h} \frac{\partial_r \sqrt{G_{\tau\tau}}}{\sqrt{g_{rr}}} . \tag{A12}$$

In Eq. (A12) the partial derivative should be taken before the limit because of Eq. (A5). In the integral of Eq. (2.24) we need the expression of Eq. (A12) for the surface gravity of a four-dimensional rotating black hole.

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